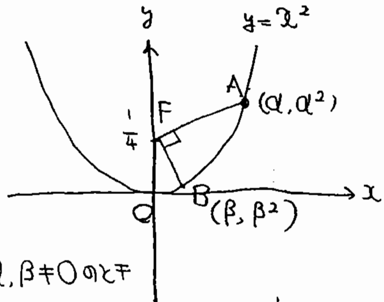
第 3 問

(1)



(i) $\alpha,\beta\neq 0$ のとき

FAの傾き $\dfrac{\alpha^2-\frac{1}{4}}{\alpha}$

FBの傾き $\dfrac{\beta^2-\frac{1}{4}}{\beta}$

傾きの積 $=-1$ ゆえ $\dfrac{\alpha^2\beta^2-\frac{1}{4}(\alpha^2+\beta^2)+\frac{1}{16}}{\alpha\beta}=-1$

$\therefore\ \alpha^2\beta^2-\frac{1}{4}(\alpha^2+\beta^2)+\frac{1}{16}=-\alpha\beta$

$\therefore\ \alpha^2\beta^2-\frac{1}{4}\{(\alpha-\beta)^2+2\alpha\beta\}+\frac{1}{16}=-\alpha\beta$

$u=\beta-\alpha$, $v=\alpha\beta$ を代入して

$v^2-\frac{1}{4}(u^2+2v)+\frac{1}{16}=-v$

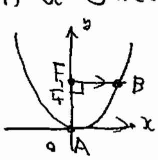
$\therefore\ 16v^2-4u^2-8v+1=-16v$

$\therefore\ 16v^2-4u^2+8v+1=0$

$\therefore\ (4v+1)^2=4u^2$

$\therefore\ \begin{cases}2u=4v+1 \\ \text{又は} \\ 2u=-4v-1\end{cases}$ $(v\neq 0)$ ⟺ $\begin{cases}v=\frac{u}{2}-\frac{1}{4} \\ v=-\frac{u}{2}-\frac{1}{4}\end{cases}$

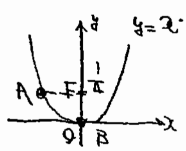
(ii) $\alpha=0$ のとき $\beta^2=\frac{1}{4}$ より $\beta=\frac{1}{2}$



$\therefore\ u=\beta-\alpha=\frac{1}{2}$

$v=\alpha\beta=0$

(iii) $\beta=0$ のとき



$\alpha^2=\frac{1}{4}$ $(\alpha<0)$ から $\alpha=-\frac{1}{2}$

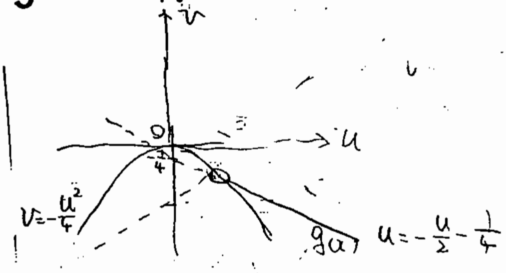
$\begin{cases}u=\beta-\alpha=\frac{1}{2} \\ v=0\end{cases}$

また α,β の存在条件として

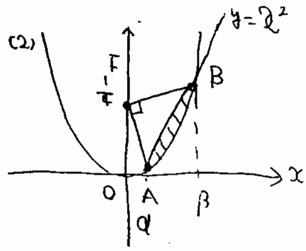
2次方程式 $t^2-(\beta-\alpha)t-\alpha\beta=0$ ⟺ $t^2-ut-v=0$ が異なる2解をもてば α,β は存在するので

$\alpha,-\beta$ が存在 $u^2+4v>0$ $\therefore\ v>-\frac{1}{4}u^2$

また $u>0$



(2)



分割して考える

$\triangle FAB=\sqrt{\alpha^2+(\alpha^2-\frac{1}{4})^2}\times\sqrt{\beta^2+(\beta^2-\frac{1}{4})^2}\times\frac{1}{2}$

$=\sqrt{(\alpha^2+\frac{1}{4})^2}\times\sqrt{(\beta^2+\frac{1}{4})^2}\times\frac{1}{2}$

$=(\alpha^2+\frac{1}{4})(\beta^2+\frac{1}{4})\times\frac{1}{2}$

$=\int_\alpha^\beta(g(x)-x^2)dx=\int_\alpha^\beta(x-\alpha)(\beta-x)$

(直線ABの式を $g(x)$ とおく)

$=\dfrac{(\beta-\alpha)^3}{6}$ (積分の公式より)

よって $S=(\alpha^2+\frac{1}{4})(\beta^2+\frac{1}{4})\cdot\frac{1}{2}+\dfrac{(\beta-\alpha)^3}{6}$

$=\dfrac{\alpha^2\beta^2}{2}+\frac{1}{8}\{(\alpha-\beta)^2+2\alpha\beta\}+\frac{1}{32}+\dfrac{(\beta-\alpha)^3}{6}$

$=\dfrac{v^2}{2}+\frac{1}{8}(u^2+2u)+\frac{1}{32}+\dfrac{u^3}{6}$

$=\dfrac{u^2-\frac{1}{2}u+\frac{1}{16}}{2}+\frac{1}{8}(u^2+2u)+\dfrac{u^3}{6}+\frac{1}{32}$

$=\dfrac{u^3}{6}+\frac{5}{8}u^2+\frac{1}{16}=f(u)$ とおく

$f'(u)=\dfrac{u^2}{2}+\frac{5}{4}u=\frac{u}{4}(2u+5)$

u	0	
$f'(u)$	+	
$f(u)$	↗	

よって $f(0)<S$

$\therefore\ \frac{1}{16}<S$

3 点数	14